U = mean velocity, L/t
u, v = point velocity
u^* = friction velocity, $\sqrt{\tau_w/\rho}$
u^+ = u/u^*
u_0 = velocity at average stream cross section h_0
$\bar{u}$ = average velocity of film of thickness h
X = parameter as defined by Equation (29)
x, y = direction coordinate, L
y^+ = dimensionless distance, $yu^*\rho/\mu$
y_1 = distance from the inner wall
y_2 = distance from the outer wall, L

Greek Letters

α = r_1/r_2, α_1 r_i/r_2
μ = coefficient of viscosity, m/Lt
ν = kinematic viscosity, L^2/t
ρ = density
ρ_g = density of air
ρ_L = density of water, m/L^3
σ = coefficient of surface tension of water, F/L
τ = shear stress
τ_1 = shear stress on the inner wall
τ_2 = shear stress on the outer wall
τ_i = shear stress at air–water interface, F/L^2
ϕ = function as defined by Equations (26) and (27)
Φ = parameter as defined by Equation (28)
λ = r_m/r_2

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A Method for the Noninteracting Control of a Class of Linear Multivariable Systems

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A method is presented by which a particular class of interacting linear multivariable systems may be decoupled into independent, first-order subsystems containing a single output as a function of a single manipulatable input and a single measurable input. This is accomplished by application of compensators in the form of feedforward amplifiers and proportional plus derivative feedback controllers. Simultaneous application of a stabilizing feedback controller and a feedforward amplifier to each decoupled subsystem results in perfect control of system outputs $y_i(t)$, that is, $y_i(t) = 0$ for all $t \geqq 0$. In addition, the compensating device contains degrees of freedom that make it possible to set output forms within any limits desired in the case that subsystem feedforward controllers operate imperfectly, and obtain better fit between the linear system model, which forms the basis for controller design, and the corresponding nonlinear system model.

A physical system is defined as multivariable when it has a multiplicity of inputs and outputs. The most important characteristic of such systems is that they will generally be interacting or cross coupled, a condition that occurs when there are inputs which simultaneously affect more than one output. Kavanagh (4, 5) was the first to apply rigorously matrix methods to the analysis of such systems and his work is a framework for a basic understanding of the properties of multivariable control systems.

If conventional control is applied to any output y_i of a multivariable system, it will not generally be possible to eliminate the effect of other controller signals or external upsets on y_i. These cross effects may, in addition, be of sufficient magnitude to make it impossible to control y_i

within desired limits. Under these circumstances it becomes desirable to eliminate cross effects by means of some additional control device.

Some early attempts to decouple linear multivariable systems (*1* to *3*, *6*, *7*) have the drawback that the compensating devices become extremely complex for systems with a large number of outputs. More recent methods (*11*, *12*) utilize state-variable feedback and make it possible to decouple the linear multivariable system into first-order independent subsystems by means of feedback and feedforward amplifiers. The method is limited to n input, n output servomechanisms.

Mesarovic (*8*, *9*) outlines a simple method for achieving noninteraction by first transforming the linear multivariable system to the so-called V canonical form so that the cross effects to be eliminated become internal feedback signals. These signals may be immediately eliminated by addition of corresponding external feedback loops. Mesarovic applies this technique only to n input, n output servomechanisms. The development to follow will be based on a modified form of the V canonical transformation, referred to as the V' canonical structure, which may be used for systems with an unequal number of inputs and outputs.

MATHEMATICAL DESCRIPTION OF THE SYSTEM

The n input, m output linear multivariables system to be studied is represented by

$$\frac{d\bar{y}_i(t)}{dt} = \sum_{j=1}^{n} K_{ij}\bar{x}_j(t) + \sum_{k=1}^{m} N_{ik}\bar{y}_k(t) \quad (i = 1,2,\ldots,m) \tag{1}$$

The following restrictions are imposed on the system of Equation (1): (a) all system parameters are known and invariant, (b) all inputs are either manipulatable or measurable, (c) the number of manipulatable inputs is equal to or greater than the number of outputs.

Restriction (c) is imposed so that control of each output is possible. Upon addition of controllers to the m manipulatable inputs $\bar{x}_1(t), \bar{x}_2(t),\ldots,\bar{x}_m(t)$ required for control of the m outputs $\bar{y}_1(t), \bar{y}_2(t),\ldots \bar{y}_m(t)$ each set point is set $= 0$ and output deviations may arise only if there are additional external inputs $\bar{x}_{m+1}(t), \bar{x}_{m+2}(t), \ldots, \bar{x}_n(t)$. Therefore only systems for which $n \geqq m$ will be considered.

The Laplace transformation of Equation (1) yields

$$s\bar{y}_i(s) = \sum_{j=1}^{n} K_{ij}\bar{x}_j(s) + \sum_{k=1}^{m} N_{ik}\bar{y}_k(s) \quad (i = 1,2,\ldots, m) \tag{2}$$

For the work to follow, $\bar{y}_i(s)$ is represented by y_i and $\bar{x}_i(s)$ by x_i for convenience. Using this notation, we can represent the m Equations (2) in the matrix form

$$R_{mxm}Y_{mx1} = K_{mxn}X_{nx1} + N_{mxm}Y_{mx1} \tag{3}$$

Solving

$$Y_{mx1} = [\,[R_{mxm} - N_{mxm}]^{-1} K_{mxn}]\, X_{nx1} \tag{4}$$

or

$$Y_{mx1} = P_{mxn}X_{nx1} \tag{5}$$

where $P_{mxn} = [\,[R_{mxm} - N_{mxm}]^{-1} K_{mxn}]$, the matrix of transfer functions relating the y_i to each x_i.

DEVELOPMENT OF THE METHOD

The first step in the decoupling procedure is to transform the system to the V' canonical structure. In this form, cross effects are converted to internal feedback signals which may be eliminated identically by means of corresponding external feedback controllers. Specifically, the first step in the transformation, with respect to a given output $y_i (i = 1, 2, \ldots, m)$, is to convert the inputs $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m, x_{m+2}, \ldots, x_n$ to internal feedback loops with the ultimate aim to make y_i a function only of the manipulatable input x_i and the measurable input x_{m+1}. It will be required that the same measurable input x_{m+1} be included in all m subsystems formed as above. This will make it possible to consider the limiting m by n system for which $n = m + 1$, as well as systems for which $n > m + 1$.

The above technique requires that there be an output corresponding to each input to be eliminated. Since the number of inputs is greater than the number of outputs, "virtual" outputs $y_{m+1}, y_{m+2}, \ldots, y_n$ must be introduced by the equations

$$\begin{aligned} Y_{m+1} &= K_{m+1,1}x_1 + K_{m+1,2} + \cdots + K_{m+1,n}x_n \\ &\vdots \\ Y_n &= K_{n1}x_1 + K_{n2}x_2 + \cdots + K_{nn}x_n \end{aligned} \tag{6}$$

where the K_{ij} are arbitrary constants relating virtual outputs to system inputs. This is the second step in the transformation. Since Equation (6) is added to the transformed system, it must also be added to the original system of Equation (5). This is valid since such addition in no way alters the behavior of the original P canonical system.

The V' canonical structure thus formed is represented mathematically by

$$Y_{nx1} = F_{nxn}X_{nx1} + F'_{nxn}V_{nxn}Y_{nx1} \tag{7}$$

In order to realize subsystems of the desired form, compensating controllers must be applied to the system represented by Equation (7) that will cancel out the term $F'_{nxn}V_{nxn}Y_{nx1}$, leaving

$$Y_{nx1} = F_{nxn}X_{nx1} \tag{8}$$

which, in expanded matrix form, becomes

$$\begin{pmatrix} y_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} F_{11} & & & F_{1,m+1} & \\ & \ddots & 0 & \vdots & 0 \\ & 0 & \ddots & \vdots & \\ & & F_{mm} & F_{m,n+1} & \\ K_{m+1,1} & \cdots & \cdots & \cdots & K_{m+1,n} \\ K_{n1} & \cdots & \cdots & \cdots & K_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{pmatrix}$$

From the above matrix form the desired subsystem equations may be written as

$$\begin{aligned} y_1 &= F_{11}x_1 + F_{1,m+1}x_{m+1} \\ y_2 &= F_{22}x_2 + F_{2,m+1}x_{m+1} \\ &\vdots \\ y_m &= F_{mm}x_m + F_{m,m+1}x_{m+1} \end{aligned} \tag{9}$$

The realization of Equations (9) from Equation (7) is immediately accomplished by adding a matrix of feedback compensating controllers C_{nxn} such that $C_{nxn} = -V_{nxn}$, which is equivalent to making $C_{ij} = -V_{ij}$ for each i and j. The addition of the compensating devices to the system of Equation (7) may be given mathematically by

$$Y_{nx1} = F_{nxn}X_{nx1} + F'_{nxn}V_{nxn}Y_{nx1} + F'_{nxn}C_{nxn}Y_{nx1} \quad (10)$$

$$Y_{nx1} = F_{nxn}X_{nx1} + F'_{nxn}V_{nxn}Y_{nx1} + F'_{nxn}(V_{nxn})Y_{nx1}$$

$$Y_{nx1} = F_{nxn}X_{nx1} \quad (11)$$

Equation (11) is seen to be identical to Equation (8). The compensation technique of Equation (10) for any given y_i may be represented by

$$y_i = F_{ii}x_i + F_{i,m+1}x_{m+1} + F_{ii}[V_{i1}y_1 + V_{i2}y_2 + \ldots + V_{i,i-1}y_{i-1} + V_{i,i+1}y_{i+1} + \ldots + V_{im}y_m + V_{i,m+2}y_{m+2} + \ldots + V_{in}y_n] + F_{ii}[C_{i1}y_1 + C_{i2}y_2 + \ldots + C_{i,i-1}y_{i-1} + C_{i,i+1}y_{i+1} + \ldots + C_{im}y_m + C_{i,m+2}y_{m+2} + \ldots + C_{in}y_n] \quad (12)$$

Once the system has been decoupled into the m subsystems given by Equations (9), final control elements may be added. A feedback controller is added to stabilize each subsystem and a feedforward controller is added by which perfect control may be achieved. The final control elements for any subsystem i operate independently of the rest of the subsystems and may be designed so as to optimize the form of the corresponding output y_i without having to consider the effect of system interactions, since such interactions no longer exist. The configuration of any subsystem i, upon addition of final control elements, is shown in Figure 1. For this subsystem, y_i is represented by

$$y_i = \frac{[F_{i,m+1} + F_{ii}G_{ffi}]}{1 + F_{ii}G_{fbi}} x_{m+1} \quad (13)$$

Perfect output control for any x_{m+1} ($y_i = 0$ for all $t \geqq 0$) can now be accomplished by requiring that

$$G_{ffi} = \frac{F_{i,m+1}}{F_{ii}} \quad (14)$$

The subsystems of Equations (9), which are only partially decoupled, are of simple enough form that the feedforward controllers could be omitted and still make it possible to achieve good output control. The relation between y_i and x_{m+1}, analogous to Equation (13), would then be

$$y_i = \frac{F_{i,m+1}}{1 + F_{ii}G_{fbi}} x_{m+1} \quad (15)$$

In either form stability analysis is identical. The value of partial decoupling is that by omission of the G_{ffi} a considerable saving of controllers is possible while still making it possible to obtain good output control with simple subsystem form in either case.

The purpose of the development thus far has been to outline the overall advantages of the V' canonical form, the configuration and general form of the compensating controllers, the method of adding final control elements to the resulting independent subsystems, and the general form of the overall subsystem transfer functions. F_{ij} and V_{ij} (and hence the C_{ij}) of the V' canonical structure will

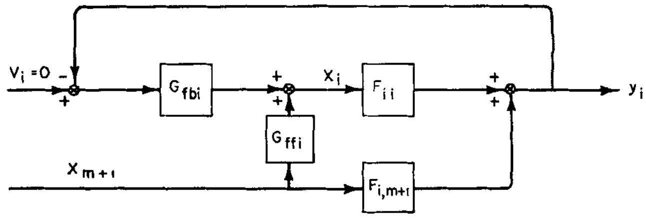

Fig. 1. Configuration of typical subsystem.

now be evaluated in terms of the P_{ij} of the original P canonical structure. The two structures must be mathematically equivalent, so Equation (5) plus Equation (6) must be equated to Equation (7):

$$Y_{nx1} = P_{nxn}X_{nx1} = F_{nxn}X_{nx1} + F'_{nxn}V_{nxn}Y_{nx1} \quad (16)$$

$$Y_{nx1} = F_{nxn}P^{-1}{}_{nxn} + F'_{nxn}V_{nxn}Y_{nx1} \quad (17)$$

$$I_{nxn} = F_{nxn}P^{-1}{}_{nxn} + F'_{nxn}V_{nxn} \quad (18)$$

The form of Equation (18) is particularly convenient for evaluation of the F_{ij} and V_{ij}, since the equivalent form $P_{nxn} = F_{nxn} + F'_{nxn}V_{nxn}P_{nxn}$ contains the term $[R_{nxn} - N_{nxn}]^{-1}$, which would lead to high-order terms in s and complicate the solution. From a practical standpoint, then, it will be required that $P^{-1}{}_{nxn}$ exist. This is equivalent to requiring that $K^{-1}{}_{nxn}$ exist since, from Equations (5) plus (6)

$$P^{-1}{}_{nxn} = K^{-1}{}_{nxn}[R_{nxn} - N_{nxn}] \quad (19)$$

Denoting $P^{-1}{}_{nxn}$ in Equation (18) by T_{nxn}, we get

$$I_{nxn} = F_{nxn}T_{nxn} + F'_{nxn}V_{nxn} \quad (20)$$

Equating each element of the identity matrix on the left of Equation (20) to the corresponding element on the right of Equation (20) with respect to any row i ($i = 1, 2, \ldots, m$), we obtain

$$1 = F_{ii}T_{ii} + F_{i,m+1}T_{m+1,i} \qquad (i = 1, 2, \ldots, m) \quad (21)$$

$$0 = F_{ii}T_{i,m+1} + F_{i,m+1}T_{m+1,m+1} \qquad (i = 1, 2, \ldots, m) \quad (22)$$

$$0 = F_{ii}T_{ij} + F_{i,m+1}T_{m+1,j} + F_{ii}V_{ij} \qquad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n; j \neq i, m+1) \quad (23)$$

Simultaneous solution of Equations (21) to (23) yields

$$F_{ii} = \frac{T_{m+1,m+1}}{T_{ii}T_{m+1,m+1} - T_{m+1,i}T_{i,m+1}} \quad (24)$$

$$F_{i,m+1} = \frac{-T_{i,m+1}}{T_{ii}T_{m+1,m+1} - T_{m+1,i}T_{i,m+1}} \quad (25)$$

$$V_{ij} = \frac{T_{i,m+1}T_{m+1,j} - T_{ij}T_{m+1,m+1}}{T_{m+1,m+1}} \quad (26)$$

$$(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n;\ j \neq i, m+1)$$

Next, the form of the T_{ij} must be determined. From Equations (5) plus (6)

$$P_{nxn} = [R_{nxn} - N_{nxn}]^{-1} K_{nxn}$$

$$T_{nxn} = P^{-1}{}_{nxn} = K^{-1}{}_{nxn}[R_{nxn} - N_{nxn}]$$

$$T_{nxn} = \frac{J'_{nxn}}{|K_{nxn}|}[R_{nxn} - N_{nxn}] \quad (27)$$

To evaluate completely F_{ii}, $F_{i,m+1}$ and V_{ij} from Equations

(24) to (26), we must evaluate the expressions for the T_{ij} from Equation (27); this yields

$$T_{m+1,m+1} = \frac{J_{m+1,m+1}}{|K_{nxn}|} \quad (28)$$

$$T_{i,m+1} = \frac{J_{m+1,i}}{|K_{nxn}|} \quad (29)$$

$$T_{ii} = \frac{1}{|K_{nxn}|}[-J_{1i}N_{1i} - J_{2i}N_{2i} - \cdots\cdots - J_{i-1,i}N_{i-1,i}$$

$$+ J_{ii}R_{ii} - J_{i+1,i}N_{i+1,i} - \cdots\cdots - J_{mi}N_{mi}] \quad (30)$$

$$\underset{(j>m)}{T_{ij}} = \frac{J_{ji}}{|K_{nxn}|} \quad (31)$$

$$T_{m+1,i} = \frac{1}{|K_{nxn}|}[-J_{1,m+1}N_{1i} - J_{2,m+1}N_{2i} - \cdots\cdots - J_{i-1,m+1}N_{i-1,i} + J_{i,m+1}R_{ii} - J_{i+1,m+1}N_{i+1,i} - \cdots\cdots - J_{m,m+1}N_{mi}] \quad (32)$$

$$\underset{(j>m)}{T_{m+1,j}} = \frac{J_{j,m+1}}{|K_{nxn}|} \quad (33)$$

The expressions $T_{ii}T_{m+1,m+1} - T_{i,m+1}T_{m+1,i}$; $T_{i,m+1}T_{m+1,j} - T_{ij}T_{m+1,m+1}$ $(j \leq m)$; and $T_{i,m+1}T_{m+1,j} - T_{ij}T_{m+1,m+1}$ $(j > m)$ are evaluated from Equations (28) to (33) to give

$$T_{ii}T_{m+1,m+1} - T_{i,m+1}T_{m+1,i} = \frac{1}{|K_{nxn}|^2}\sum_{\substack{k=1\\k\neq i}}^{m} N_{ki}[J_{m+1,i}J_{k,m+1} - J_{m+1,m+1}J_{ki}] + R_{ii}[J_{m+1,m+1}J_{ii} - J_{m+1,i}J_{i,m+1}] \quad (34)$$

$$\underset{(j\leq m)}{T_{i,m+1}T_{m+1,j} - T_{ij}T_{m+1,m+1}} = \frac{1}{|K_{nxn}|^2}\sum_{\substack{k=1\\k\neq i}}^{m} N_{kj}[J_{m+1,m+1}J_{ki} - J_{m+1,i}J_{k,m+1}] + R_{jj}[J_{m+1,i}J_{j,m+1} - J_{m+1,m+1}J_{ji}] \quad (35)$$

$$T_{i,m+1}T_{m+1,j} - T_{ij}T_{m+1,m+1} = \frac{J_{m+1,i}J_{j,m+1} - J_{ji}J_{m+1,m+1}}{|K_{nxn}|^2} \quad (36)$$

Equations (34) to (36) may be considerably simplified by introducing a general rule relating to determinants given by Muir (*10*). This rule may be written in the two equivalent forms:

1. $J_{m+1,m+1}J_{ki} - J_{k,m+1}J_{m+1,i} = (-1)^{2m+2+k+i}M_{ki}|K_{nxn}|$ (37)
2. $J_{k,m+1}J_{m+1,i} - J_{m+1,m+1}J_{ki} = (-1)^{2m+3+k+i}M_{ki}|K_{nxn}|$

Insertion of Equations (37) into Equations (34) to (36) and combination with Equations (24) to (26) yield

$$F_{ii} = \frac{J_{m+1,m+1}}{[(-1)^{2(m+i+1)}M_{ii}]s + \left[\sum_{\substack{k=1\\k\neq i}}^{m}(-1)^{2m+3+k+i}M_{ki}N_{ki} - (-1)^{2(m+i+1)}N_{ii}M_{ii}\right]} \quad (38)$$

$$F_{i,m+1} = \frac{-J_{m+1,i}}{[(-1)^{2(m+i+1)}M_{ii}]s + \left[\sum_{\substack{k=1\\k\neq i}}^{m}(-1)^{2m+3+k+i}M_{ki}N_{ki} - (-1)^{2(m+i+1)}N_{ii}M_{ii}\right]} \quad (39)$$

$$\underset{(j\leq m)}{V_{ij}} = \frac{[(-1)^{2m+i+j+3}M_{ji}]s + \left[\sum_{\substack{k=1\\k\neq j}}^{m}(-1)^{2m+2+k+i}N_{kj}M_{ki} - (-1)^{2m+i+j+3}N_{jj}M_{ji}\right]}{J_{m+1,m+1}} \quad (40)$$

$$\underset{(j>m)}{V_{ij}} = \frac{(-1)^{2m+3+j+i}M_{ji}}{J_{m+1,m+1}} \quad (41)$$

Equations (38) to (41) represent the final form of the F_{ij} and V_{ij} for the V' canonical structure. Recognizing that, for any given system the J_{ij}, N_{ij}, and M_{ij} will be fixed, we can rewrite Equations (38) to (41) in the compact form

$$F_{ii} = \frac{A_{ij}}{B_{ij}s + S_{ij}} \quad (42)$$

$$F_{i,m+1} = \frac{D_{ij}}{B_{ij}s + S_{ij}} \quad (43)$$

$$\underset{(j\leq m)}{V_{ij}} = E_{ij}s + U_{ij} \quad (44)$$

$$\underset{(j>m)}{V_{ij}} = H_{ij} \quad (45)$$

The following properties characterize any order system upon transformation into the V' canonical structure as may be deduced from Equations (42) to (45) above:

1. The subsystem transfer functions F_{ij} are always of first order.
2. The transfer functions V_{ij} $(j \leq m)$ and hence the compensating controllers C_{ij} $(j \leq m)$ are first-order polynomials in *s*. From Equation (12) it can be seen that each C_{ij} is associated with a corresponding y_j. The signal transmitted from each compensating controller C_{ij} to the corresponding subsystem *i* is thus $C_{ij}y_j = E_{ij}\,s\,y_j + U_{ij}y_j$. This signal, upon inversion to the time domain, becomes

$$C_{ij}y_j(t) = E_{ij}\frac{dy_j(t)}{dt} + U_{ij}y_j(t)$$

Therefore, the C_{ij} $(j \leq m)$ may be considered to be proportional plus derivative controllers and as such are physically realizable, at least to a good approximation.

3. The transfer functions V_{ij} $(j > m)$ and hence C_{ij} $(j > m)$ are constants and, by an analysis similar to that of 2 are seen to be proportional controllers or, simply, amplifiers and are physically realizable.

It is now possible to investigate in detail the form of the overall subsystem transfer function described in Equation (13) for a given feedback controller G_{fbi}. For purposes of generality, the following three-mode controller G_{fbi} will be used:

$$G_{fbi} = K_{ci}\left[1 + T_{Di}\, s + \frac{1}{T_i s} \right] \tag{46}$$

Insertion of Equations (46), (42), and (43) into Equation (13) yields for any decoupled subsystem

$$y_i = \frac{T_i s [D_{ij} + A_{ij} G_{ffi}]\, x_{m+1}}{[T_i B_{ij} + A_{ij} K_{ci} T_i T_{Di}]\, s^2 + [T_i s_{ij} + A_{ij} K_{ci} T_i]\, s + [A_{ij} K_{ci}]} \tag{47}$$

Perfect output control for any x_{m+1} ($y_i = 0$ for all $t \geqq 0$) can be accomplished by setting

$$G_{ffi} = \left(\frac{-D_{ij}}{A_{ij}} \right) \tag{48}$$

The significance of Equations (47) and (48), in terms of practical application to any order linear multivariable system, may be summarized as follows:

1. The denominator of the overall transfer function of each decoupled subsystem has a maximum order of two. Techniques for stability analysis and inversion to the time domain of second-order transfer functions are relatively simple and well known.

2. The feedforward controller required for perfect control of each subsystem output is a simple amplifier. These feedforward controllers may be considered as additional decoupling controllers for the purpose of eliminating the effect of x_{m+1} on each subsystem.

The final functional form of the F_{ij} and overall transfer functions for each decoupled subsystem as well as the decoupling controllers C_{ij} and G_{ffi} has now been obtained in terms of system parameters. The following general properties of the decoupled system in the V' canonical form with final control elements will now be investigated: (1) physical significance of the arbitrary K_{ij} introduced in Equation (6); (2) number of compensating controllers required for complete noninteraction; (3) degrees of freedom within the compensated system and their significance with respect to controller design.

The physical significance of the arbitrary K_{ij} in Equation (6) may best be understood by investigating the system in the P canonical form. Since it has been postulated that the V' canonical form and P canonical form plus virtual outputs be completely equivalent, the compensating controllers C_{ij} may be applied to the system in either form. Application of the nonzero portion of the matrix C_{nxn} to the P canonical system of Equation (5) and virtual system of Equation (6) yields the system shown in Figure 2, before addition of the final controllers G_{fbi} and G_{ffi}. This diagram clearly shows that the matrix $K_{(n-m)xn}$ of arbitrary K_{ij} is actually a portion of the compensating control device and so each arbitrary K_{ij} represents a separate feedforward compensating controller in the form of an amplifier. These controllers will be referred to as *forward compensators.* The virtual outputs, in turn, represent intermediate controller signals forming a portion of the input to the nonzero portion of the matrix C_{nxn} of compensating controllers. The original matrix form of C_{nxn} shows that the virtual output y_{m+1} is never fed back to the real portion of the system and may be discarded. The corresponding forward compensators $K_{m+1,j}$ ($j = 1, 2, \ldots, n$) may be discarded. An investigation of Equation (6) shows, therefore, that there are $n(n-m-1)$ forward compensators which are actually included in the compensating control system in addition to the C_{ij}.

The total number of compensating controllers may now be determined. These are composed of the $n(n-m-1)$ forward compensators, the $m(n-2)$ C_{ij} within the matrix C_{nxn}, and the m feedforward controllers added to each of the m decoupled subsystems corresponding to the real outputs. Thus

Number of compensating controllers

$$= n(n-m-1) + m(n-2) + m$$
$$= n(n-1) - m \tag{49}$$

It is now possible to determine the number of degrees of freedom within the decoupled system in the V' canonical form with final control elements. These degrees of freedom will be equal to the number of undetermined constants within the system minus the number of restrictions imposed by setting certain output characteristics.

The undetermined system constants arise from addition of the forward compensators to the basic system and from addition of the m feedback controllers required to stabilize the m subsystems. Each of the stabilizing feedback controllers may contain as many as three undetermined constants (K_{ci}, T_{di}, or T_i). Therefore

Number of undetermined constants

$$= n(n-m-1) + 3m \tag{50}$$

Although a feedforward controller has been added to each subsystem to obtain zero $y_i(t)$ for all t, it is likely that small imperfections in the operation of system controllers could cause nonzero outputs. There is also the

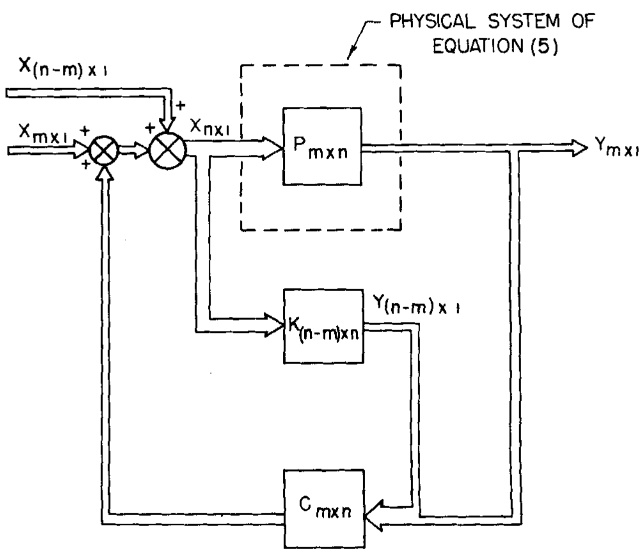

Fig. 2. Graphical presentation showing physical significance of system parameters.

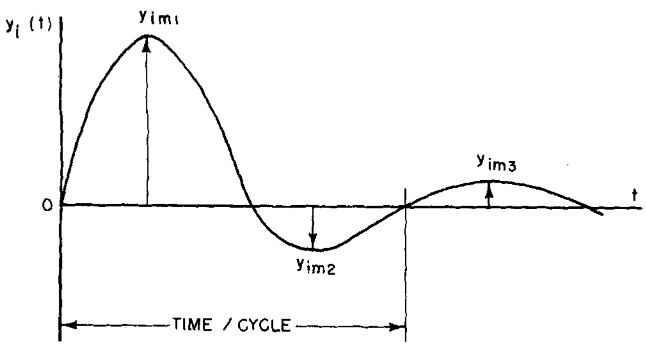

Fig. 3. Desired damped oscillatory output.

case where these feedforward controllers are omitted for reasons of economy. In these cases, output fluctuations will be limited to a damped oscillatory form in the time domain for step upsets, as shown in Figure 3. This output form is characterized by the parameters ξ_i and τ_i which affect time per cycle and the ratio y_{im1}/y_{im2} of two subsequent peaks. τ_i and ξ_i, which are referred to as the characteristic time and the damping ratio, respectively, will be set for each output $y_i(t)$ in accordance with desired output form. For the m real outputs $y_1(t)$, $y_2(t)$, $\ldots$, $y_m(t)$ this imposes $2m$ restrictions. Subtracting this number from Equation (50), we get

$$\text{Degrees of freedom} = n(n - m - 1) + 3m - 2m$$
$$= (n - m)\ (n - 1) \qquad (51)$$

The damped oscillatory form for each output $y_i(t)$ may be postulated because the overall transfer function of each decoupled subsystem is second order. Equation (47) may be rewritten as

$$y_i = \frac{\dfrac{T_i s}{A_{ij}K_{ci}}\,[D_{ij} + A_{ij}G_{ffi}]}{T_i^2 s^2 + 2\,T_i \xi_i s + 1}\, x_{m+1} \qquad (52)$$

where

$$T_i^2 = \frac{T_i B_{ij} + A_{ij}K_{ci}T_i T_{Di}}{A_{ij}K_{ci}} \qquad (53)$$

$$2\,T_i\xi_i = \frac{T_i S_{ij} + A_{ij}K_{ci}T_i}{A_{ij}K_{ci}} \qquad (54)$$

For a step input x_{m+1}, inversion of Equation (52) to the time domain yields the damped oscillatory form desired of $y_i(t)$. The restrictions τ_i and ξ_i are functions of K_{ci}, T_{di}, and T_i as well as the forward compensating K_{ij} which are contained in A_{ij}, B_{ij}, and S_{ij}. Setting τ_i and ξ_i of each subsystem to desired values as functions of these undetermined constants yields the number of degrees of freedom given by Equation (51). The degrees of freedom existing in the system have a definite value. In particular, they give: (1) the ability to adjust the output peak amplitudes to small values, and (2) the ability to adjust manipulatable system inputs so that better correspondence is obtained between the linear model, which forms the basis for the design of the controllers, and the true nonlinear model.

CONCLUSIONS

A method has been developed by which a linear multivariable system represented by a set of first-order differential equations may be decoupled into a set of subsystems characterized by first-order transfer functions. These subsystems contain a single output as a function of a single manipulatable and a single measurable input. Decoupling is achieved by means of feedforward amplifiers and feedback proportional plus derivative controllers. This result is independent of the number of differential equations. Perfect output control is possible by addition of a feedback controller and a feedforward controller to each subsystem.

The technique is applicable to multivariable lumped parameter systems of interest to chemical engineering, such as the continuous flow, stirred-tank reactor. A particular example of such a system will be considered in a later paper.

NOTATION

$A_{ij}, B_{ij}, D_{ij}, E_{ij}, H_{ij}, S_{ij}, U_{ij}$ = constants composed of groupings of terms in Equations (38) through (41)
$C_{ij}, F_{ij}, J_{ij}, K_{ij}, N_{ij}, R_{ij}, T_{ij}, V_{ij}$ = components of corresponding matrices, as defined below
C_{nxn} = matrix of feedback compensating controllers
F_{nxn} = matrix of transfer functions in V' form
G_{fbi} = feedback control for subsystem i
G_{ffi} = feedforward controller for subsystem i
i, j, k = indices for summation
I_{nxn} = identity matrix
J_{nxn} = matrix formed by replacing each term K_{ij} of matrix K_{nxn} by its corresponding cofactor J_{ij}
J'_{nxn} = transpose of matrix J_{nxn}
K_{nxn} = matrix of constants in original differential equations
K_{ci} = proportional sensitivity
$K_{(n-m)xh}$ = matrix of arbitrary constants introduced by definition of virtual outputs
m = number of outputs
M_{ki} = determinant of the matrix formed by crossing out the rows and columns containing $K_{m+1,i}$; $K_{k,m+1}$; $K_{m+1,m+1}$; and K_{ki} in the matrix K_{nxn}
n = number of inputs ($n \geqq m$)
N_{mxm} = second matrix of constants in original differential equations
P_{mxn} = matrix of transfer functions in P form
R_{mxm} = diagonal matrix of $R_{ii} \equiv S - N_{ii}$
s = Laplace transform variable
t = time
T_{Di} = derivative time
T_i = integral time
T_{nxn} = transpose of matrix P_{nxn}
V_{nxn} = second matrix of transfer functions in V' form
X_{mx1} = column vector of manipulatable inputs
$X_{(n-m)x1}$ = column vector of additional manipulatable or measurable inputs
$\overline{X_i}$ = $X_i(t) - X_{io}$, deviation of input X_i from steady state
Y_{mx1} = column vector of real outputs
$Y_{(n-m)x1}$ = column vector of virtual outputs
$\overline{y_i}$ = $y_i(t) - y_{io}$, deviation of output y_i from steady state
y_{im} = peak value of y_i

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An Application of Noninteracting Control to a Continuous Flow Stirred-Tank Reactor

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A general method, given in a companion paper, by which an interacting linear multivariable system may be decoupled into independent subsystems, is applied to a continuous flow stirred-tank reactor. The form of the compensating controllers required for the physical system is obtained. It is then verified numerically that noninteraction is achieved, as postulated, by simulation of the linear model with compensating control on the analog computer.

The characteristic property of a multivariable system is that each of its inputs will generally affect more than one output simultaneously. Conventional control of such an interacting system can be both difficult and inefficient in that each control device must be compromised in its design. Although primary emphasis is placed on achieving adequate control of a corresponding output, care must be taken that the controller will not adversely affect the remaining system outputs. This difficulty has been eliminated in a companion paper (*1*) by describing theoretically the design of a compensating control device of generally simple form which may be applied to linear multivariable systems of any order. The technique makes it possible to break the system down into independent subsystems containing a single output as a function of a single manipulatable input and a single measurable input. Final output control of each subsystem may then be achieved in the absence of undesirable system interactions.

It is the purpose of this paper to apply the compensating technique described in reference *1* to a particular physical system, the continuous flow stirred-tank reactor. In addition to describing the form of the compensating controllers required for the given system, it will be verified numerically by means of analog computer simulation that noninteraction is achieved as postulated.

DESCRIPTION OF THE SYSTEM

The general theory of noninteracting control that has been developed (*1*) will be applied to a continuous flow stirred-tank reaction which contains a material undergoing a reaction of the type $X \xrightarrow{k}$ products, for which rate of conversion of $X(t)$ is given by $dX(t)/dt = kX(t) = A' \, e^{-E/RT(t)} X(t)$. The system, as described by Kermode and Stevens (*3*), is pictured in Figure 1.

The unsteady state heat and mass balance characterizing the system may be written as follows:

$$\frac{dX(t)}{dt} = \frac{Q(t)}{V} [X_i(t) - X(t)] - A' \, e^{-E/RT(t)} X(t) \quad (1)$$

$$\frac{dT(t)}{dt} = \frac{Q(t)}{V} [T_i(t) - T(t)] - \frac{UAF(t) \, [T(t) - T_c]}{V_p C_p \, [F(t) + 1]} - \frac{A' \, e^{-E/RT(t)} \, \Delta H X(t)}{\rho \, C_p} \quad (2)$$

$$F(t) = \frac{2 \, Q_c(t) \, \rho_c \, C_c}{UA}$$

The system will be controlled about its unstable point at